Subjective Expected Utility

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subjective probability

what is probability?

- Relative population frequencies
- Intuitive notion of likelihood
- Likelihood revealed from choices
- Probability space: states S, events $\mathcal{S}$ and probability function $\Pr : \mathcal{S} \to [0, 1]$
  - $\Pr(S) = 1$, $\Pr(\emptyset) = 0$
  - If $A \cap B = \emptyset$ then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

von neumann morgestern

- Set of prizes X
- Set of simple lotteries $\mathcal{L}$ assigning objective probabilities to prizes
- Preferences $\succ$ over $\mathcal{L}$
- **Representation**:
  - Subjective Bernoulli utility function u over prizes

$$p \succ q \qquad \Leftrightarrow \qquad \sum_{x \in X} p(x)u(x) > \sum_{x \in X} q(x)u(x)$$

- Set of consequences X
- Set of states S (and events $\mathcal{S}$)
- Acts f map states into consequences
- Preferences $\succ$ over acts
- **Representation**:
  - Subjective Bernoulli utility function u over prizes
  - Beliefs are subjective probabilities p over states

$$f \succ g \qquad \Leftrightarrow \qquad \sum_{s \in S} p(s)u(f(s)) > \sum_{s \in S} p(s)u(g(s))$$

- p_f is be the lottery over consequences induced by act f given beliefs p

$$p_f(x) = \sum_{\{s \mid f(s)=x\}} p(s)$$

- The collection of sets of the form $\{s \mid f(s) = x\}$ is a partition of S
- Therefore

$$\begin{aligned}\sum_{s\in S} p(s)u(f(s)) &= \sum_{x\in X} \sum_{\{s \mid f(s)=x\}} p(s)u(s) \\ &= \sum_{x\in X} \left[u(x) \sum_{\{s \mid f(s)=x\}} p(s) \right] = \sum_{x\in X} p_f(x)u(x)\end{aligned}$$

outiline

- Propose reasonable postulates of “rational” behavior that allow to
  1. Derive qualitative probabilities from choices
  2. Derive a numerical representation of probabilities
  3. Guarantee that choices depend only on the induced probability distribution over consequences
  4. Obtain a subjectvie expected utility representation

qualitative probabilities

states and events

- Set of states S
- Algebra of events $\mathcal{S}$ satisfies
  - $S, \emptyset \in \mathcal{S}$
  - If $A \in \mathcal{S}$ then $A^c \in \mathcal{S}$
  - If $A, B \in \mathcal{S}$ then $A \cup B \in \mathcal{S}$
- **Example**: $S = [0, 1]$, $\mathcal{S} = \{[0, 1], \emptyset, [0, 0.25), [0.25, 1]\}$

qualitative probabilitites

- $A \rhd B$ means that "A is considered strictly more likely than B"
- What properties should it satisfy?
  - Asymmetry and negative transitivity
  - $S \rhd \emptyset$
  - $S \unrhd A \unrhd \emptyset$ for every event A
  - If $C \cap A = \emptyset$ and $C \cap B = \emptyset$ then

$$A \rhd B \quad \Leftrightarrow \quad (A \cup C) \rhd (B \cup C)$$

- In that case we say that $\rhd$ is a qualitative probability relation

quantitative representation

quantitative representation

- Given a relation $\rhd$ over events, can we find a probability function Pr such that $A \rhd B$ if and only if $\Pr(A) > \Pr(B)$?
- Necessary that $\rhd$ is be a qualitative probability relation
  - A + NT are necessary to have a functional representation
  - $Pr(S) = 1$ and $\Pr(\emptyset) = 0$ imply that we must have $S \rhd \emptyset$
  - If $C \cap A = \emptyset$ and $C \cap B = \emptyset$, then $\Pr(A \cup C) - \Pr(B \cup C) = \Pr(A) - \Pr(B)$
- Not sufficient (for a counterexample see Kreps, pp. 119)

coin tosses

- **Assumption CT** For every n, there exists a partition $\{A_1^n, A_2^n, \ldots A_{2^n}^n\}$ of S consisting of equally likely events, i.e., such that $A_i^n \equiv A_j^n$ for every i, j
- Similar to a fair coin that can be tossed arbitrarily many times
- If Pr represents $\triangleright$, then it must be the case that $\Pr(A_j^n) = 1/2^n$ (why?)

coin tosses

- Coin tosses provide bounds for the probability of other events
- For B such that $S \rhd B$, let $k_n(B)$ the the smallest number such that

$$\cup_{j=1}^{k_n(B)} A_j^n \rhd B$$

- If Pr represents $\rhd$, then it must be the case that

$$\frac{k_n(B)}{2^n} > \Pr(B) \geq \frac{k_n(B) - 1}{2^n}$$

- Taking limits as $n \to \infty$ and using the pinching theorem, it must be the case that

$$\Pr(B) = \lim_{n \to \infty} \frac{k_n(B)}{2^n} \tag{1}$$

Proposition: Suppose $\rhd$ is a qualitative probability relation and assumption CT holds, and let $\Pr(\cdot)$ be the function defined by (1) on the previous slide

(a) Pr is a probability function over $(S, \mathcal{S})$

(b) If $A \trianglerighteq B$, then $\Pr(A) \geq \Pr(B)$

(c) For every event A and number $\alpha \in (0, 1)$, there exists $B \subset A$ such that $\Pr(B) = \alpha \Pr(A)$

- Still not a complete representation as we could have $A \succ B$ but $\Pr(A) = \Pr(B)$
- We need additional technical conditions need to me imposed get a complete representation,, which are essentially continuity assumptions

revealed qualitative probabilities

non-trivial rational preferences

Axiom 1: $\succ$ is asymmetric and negatively transitive

- Let f_x denote the constant act that yields x on every state
- Extend $\succ$ to consequences by saying that $x \succ y$ if and only if $f_x \succ f_y$

Axiom 2: There exist consequences x and y such that $x \succ y$

key axiom & conditional preferences

- Given acts f and g and an event A, let $f_A g$ denote the act given by

$$[f_A g](s) = \begin{cases} f(s) & \text{if } s \in A \\ g(s) & \text{if } s \notin A \end{cases}$$

- To keep notation simple, let $x_A f$ and $f_A x$ denote the acts $(f_x)_A f$ and $f_A(f_x)$ respectively

Axiom 3: For all acts f, g, h, k and all events A, $f_A h \succ g_A h$ if and only if $f_A k \succ g_A k$

- This allows us to define preferences conditional on an event
- Say that $f \succ_A g$ if there exists an act h such that $f_A h \succ g_A h$

null events

Definition: Event A is null of $f \sim_A g$ for all acts f, g

- An event if null if the outcomes on it are irrelevant for the agent's preferences
- Intuitively, null events will correspond to 0-probability events

Proposition:

(a) S is not null

(b) An event A is null if and only if $A \equiv \emptyset$

(c) Subsets of null events are null

(d) If A and B are null, then $A \cup B$ is null

sure-thing principle

Proposition: Given acts f, g and an event A, if $f \succ_A g$ and $f \succ_{A^c} g$, then $f \succ g$

- Follows from applying Axiom 3 twice

$$f = f_A f \succ g_A f = f_{A^c} g \succ g_{A^c} g = g$$

- Can be extended to finite partitions $\{A_1, A_2, \ldots, A_n\}$ of S using mathematical induction

monotonicity

- The following axiom requires that conditional preferences respect the order of unconditional preferences

Axiom 4: For all consequences x, y and non-null events A, $x \succ y$ if and only if $x \succ_A y$

simple bets

- Suppose that $x \succ y$ then $x_A y$ is a bet that pays a good outcome (x) if A happens, and a worse outcome (y) otherwise
- The preference between $x_A y$ and $x_B y$ reveals the assessed relative likelihood between A and B
- Would you rather placing a bet on event A or on event B?
- The following axiom requires that the answer does not depend on the specific prizes

Axiom 5: If $x \succ y$ and $w \succ z$ then, for all events A, B

$$x_A y \succ x_B y \quad \Leftrightarrow \quad w_A z \succ w_B z$$

revealed qualitative probabilities

Definition: Say that A is revealed to be more likely than B, $A \rhd B$, if there exists consequences x, y such that $x \succ y$ and $x_A y \succ x_B y$

Theorem: $\rhd$ is a qualitative probability relation on $(S, \mathcal{S})$

proof sketch

- Asymmetry – If $A \rhd B$, then $x_A y \succ x_B y$ for every x and y such that $x \succ y$
- Negative transitivity – $A \not\rhd B \not\rhd C$, then $x_C y \succcurlyeq x_B y \succcurlyeq x_A y$ for every x and y such that $x \succ y$, and thus $x_C y \succcurlyeq x_A y$ for every x and y such that $x \succ y$
- If A is not null and $x \succ y$, then $x \succ_A y$ which implies that $x_A y \succ y_A y = y_\emptyset y$
- Suppose that $x \succ y$, $A \cup C = \emptyset$ and $B \cup C = \emptyset$, and $A \rhd B$
  - Let $D_1 = A \setminus B$, $D_2 = B \setminus A$, $D_3 = A \cap B$ and $D_4 = C$ and $D_5 = (A \cup B \cup C)^c$
  - $\{D_1, D_2, D_3, D_4, D_5\}$ is a partition of S
  - $A \rhd B$ implies that $(x_{D_1} x_{D_2} y_{D_3} y_{D_4} y) \succ (y_{D_1} x_{D_2} x_{D_3} y_{D_4} y)$
  - Axiom 3 then implies that $(x_{D_1} x_{D_2} y_{D_3} x_{D_4} y) \succ (y_{D_1} x_{D_2} x_{D_3} x_{D_4} y)$
  - This says that $(A \cup C) \rhd (B \cup C)$

subjective expected utility

Axiom 6: For all acts f, g such that $f \succ g$ and every consequence x there exists a finite partition $\{A_1, A_2, \ldots A_n\}$ of S such that for every A_i

$$f \succ x_{A_i} g \qquad \text{and} \qquad x_{A_i} f \succ g$$

- This is similar to assumption CT, except that the events of the partition do not have to be equally likely
- Since f, g and x are arbitrary, there have to exist arbitrarily fine partitions with events that are "almost" equally likely
- Axiom 6 also has implicit "continuity" assumptions that guarantee the existence of quantitative probabilities

Theorem:

(a) There exists a unique probability function over states such that $\Pr(A) > \Pr(B)$ if and only if $A \rhd B$

(b) Moreover, for every event A and number $\alpha \in (0, 1)$, there exists $B \subset A$ such that $\Pr(B) = \alpha \Pr(A)$

probabilistic sophistication

- p_f is be the lottery over consequences induced by act f given beliefs

$$p_f(x) = \sum_{\{s \mid f(s)=x\}} \Pr(s)$$

Proposition: For all acts f and g, if $p_f = p_g$, then $f \sim g$

- This allows to define preferences over lotteries assigning probabilities to consequences
- Given lotteries p and q, say that $p \succ q$ if there exist acts f, g such tat $p = p_f$, $q = p_g$, and $f \succ g$

Proposition: Given lotteries p, q, h and $\alpha \in (0, 1)$

$$\alpha p + (1-\alpha)h \succ \alpha q + (1-\alpha)h \qquad \Leftrightarrow \qquad p \succ q$$

- That is, the agent's preferences over lotteries satisfy vNM linearity, hence

Theorem: There exists a unique probability function over states Pr and a unitility function u unique up to positive affine transformations such that

$$f \succ g \qquad \Leftrightarrow \qquad \sum_{s \in S} \Pr(s) u(f(s)) > \sum_{s \in S} \Pr(s) u(g(s))$$